San Diego Math League High School Division, Round 2a

January 30, 2010

1. Three consecutive positive odd integers x, y, z satisfy $y^2 - x^2 = 344$ and $z^2 > y^2$. What is the value of $z^2 - y^2$?

2. How many of the divisors of 27! are larger than 26!?

3. What is the remainder when $6^{101} + 6^{102} + 6^{103} + \cdots + 6^{200}$ is divided by 43?

4. In acute triangle ABC, altitudes $\overline{AD}$ and $\overline{BE}$ meet at H. If $AD + BE = 35$, $AH = 9$, and $BH = 12$, then what is DH?

5. Evaluate
$$1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \cdots + \frac{1}{1+2+3+\cdots+2010}.$$

6. If $a^{2x} = 3$, then what is $\dfrac{a^{3x} + a^{-3x}}{a^x + a^{-x}}$?

7. In parallelogram $ABCD$, point M is the midpoint of $\overline{BC}$, and point N is the midpoint of $\overline{CD}$. Lines $\overleftrightarrow{AM}$ and $\overleftrightarrow{BN}$ intersect in P. Find AP/AM.

8. Let f be a function defined on the nonnegative integers such that $f(2n) = 3f(n)$ and $f(2n+1) = f(2n)+1$ for all nonnegative integers n. How many nonnegative integers satisfy $f(n) \leq 800$?

9. Let a, b, and c be the roots of $x^3 - 2x + 2$. Find $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}$.

10. The six faces of polyhedron $\mathcal{A}$ are congruent equilateral triangles. Each face of regular octahedron $\mathcal{B}$ is congruent to a face of $\mathcal{A}$. Find the ratio of the radius of the sphere inscribed in $\mathcal{A}$ to the radius of the sphere inscribed in $\mathcal{B}$.

11. Evaluate $\displaystyle\frac{1}{2^{2010}} \sum_{n=0}^{1005} (-3)^n \binom{2010}{2n}$.

12. David has a deck of 5 cards, numbered from 1 to 5. Six times, he shuffles the deck, selects a card at random, shouts the number on the card, and returns it to the deck. What is the expected value of the smallest number David shouts?

1. Find all ordered triples of positive integers (a, b, c) such that $a! = 4(b!) + 10(c!)$.

2. Solve the equation $\sqrt{x+3-4\sqrt{x-1}} + \sqrt{x+8-6\sqrt{x-1}} = 1 + \sqrt{2}$.

3. If $a_0, a_1, \ldots, a_{4020}$ are constants such that

$$(1+x+x^2)^{2010} = \sum_{n=0}^{4020} a_n x^n,$$

then find $a_0 + a_3 + a_6 + a_9 + \cdots + a_{4020}$.

4. How many subsets $\{a, b, c\}$ of $\{1, 2, 3, \ldots, 15\}$ satisfy $b - a \geq 3$ and $c - b \geq 3$?

5. Let ABC be a triangle and let D and E be points on $\overline{BC}$ such that $\angle DAB = \angle EAC$. If $BD = 2$, $DE = 6$, and $EC = 8$, then what is AC/AB?

6. I'm thinking of a positive integer no greater than 144. You can pick any subset of the first 144 positive integers and ask me if my number is among those integers. If I answer yes, then you pay me \$2. If I answer no, then you pay me \$1. What is the smallest number of dollars you need to make sure you can determine what my number is?